14.1. Functions of several variables

Def Consider a function $f(x,y)$ of two variables.

(1) Its level curve (or contour curve) at k is the curve on the xy-plane given by $f(x,y)=k$.

e.g. The circle $x^2+y^2=1$ is the level curve of $f(x,y)=x^2+y^2$ at 1.

The line $2x+y=3$ is the level curve of $f(x,y)=2x+y$ at 3.

(2) Its contour map is a collection of level curves.

Note The level curve at k is the cross section of the graph $z=f(x,y)$ for $z=k$.

Def For a function $f(x,y,z)$ of three variables, its level surface at k is the surface in $\mathbb{R}^3$ given by $f(x,y,z)=k$.

e.g. The sphere $x^2+y^2+z^2=4$ is the level surface of $f(x,y,z)=x^2+y^2+z^2$ at 4.

The plane $x+2y+3z=6$ is the level surface of $f(x,y,z)=x+2y+3z$ at 6.

Ex For each function, draw a contour map with 5 level curves.

(1) $g(x,y) = \sqrt{16-16x^2-y^2}$

Sol Find the domain.

$16-16x^2-y^2 \geq 0 \rightsquigarrow 16x^2+y^2 \leq 16$: inside an ellipse.

($\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $a^2x^2+b^2y^2=c^2$ defines an ellipse centered at the origin)

The level curve at k is given by $g(x,y)=k$.

In this case, all levels are nonnegative.

($g(x,y)=\sqrt{16-16x^2-y^2} \geq 0$)

$k=0$: $\sqrt{16-16x^2-y^2}=0 \rightsquigarrow 16-16x^2-y^2=0$

$\rightsquigarrow 16x^2+y^2=16$: an ellipse

$x=0 \Rightarrow y=\pm 4$, $y=0 \Rightarrow x=\pm 1$.

$k=1$: $\sqrt{16-16x^2-y^2}=1 \rightsquigarrow 16-16x^2-y^2=1$

$\rightsquigarrow 16x^2+y^2=15$: an ellipse

$x=0 \Rightarrow y=\pm\sqrt{15}$, $y=0 \Rightarrow x=\pm\sqrt{\frac{15}{16}}$.

$k=2$: $\sqrt{16-16x^2-y^2} = 2 \rightsquigarrow 16-16x^2-y^2 = 4$

$\rightsquigarrow 16x^2+y^2 = 12$: an ellipse

$x=0 \Rightarrow y = \pm\sqrt{12}$, $y=0 \Rightarrow x = \pm\sqrt{\frac{3}{4}}$.

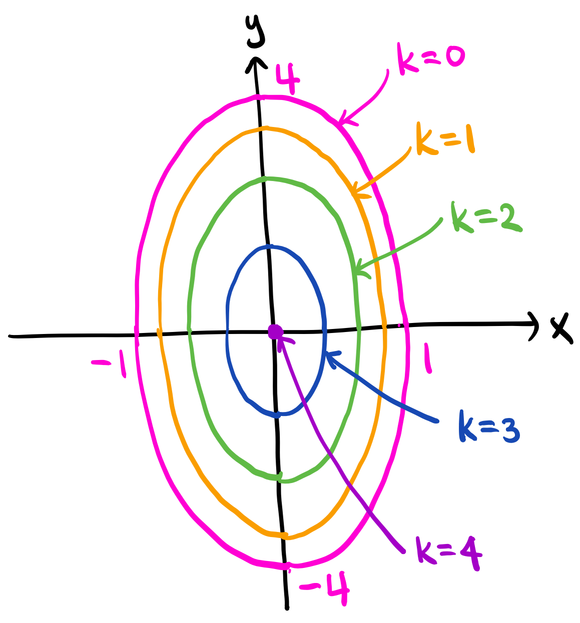
$k=3$: $\sqrt{16-16x^2-y^2} = 3 \rightsquigarrow 16-16x^2-y^2 = 9$

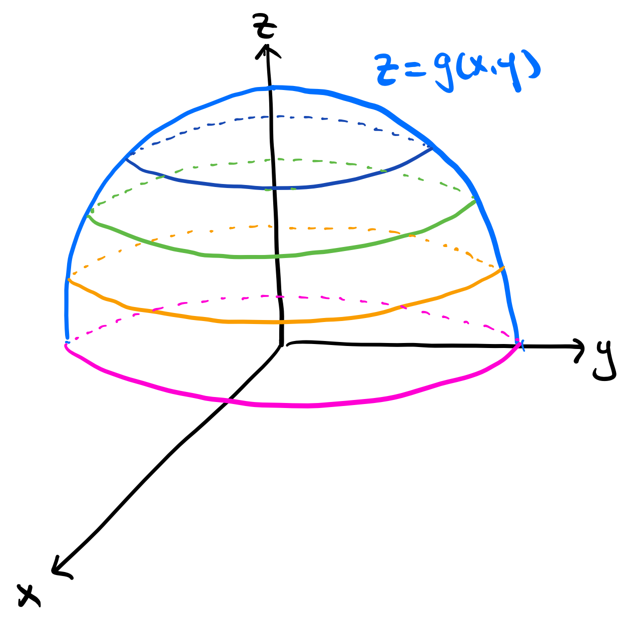
$\rightsquigarrow 16x^2+y^2 = 7$: an ellipse

$x=0 \Rightarrow y = \pm\sqrt{7}$, $y=0 \Rightarrow x = \pm\sqrt{\frac{7}{16}}$.

$k=4$: $\sqrt{16-16x^2-y^2} = 4 \rightsquigarrow 16-16x^2-y^2 = 16$

$\rightsquigarrow 16x^2+y^2 = 0 \rightsquigarrow x=y=0$: a point.





<u>Note</u> The contour map shows the projection of the graph $z=g(x,y)$.

(2) $h(x,y) = \dfrac{4y}{x^2+y^2}$

Sol Find the domain.

$x^2+y^2 \neq 0 \rightsquigarrow$ all points except $(0,0)$.

The level curve at k is given by $h(x,y)=k$.

$k=0$: $\dfrac{4y}{x^2+y^2} = 0$

$\rightsquigarrow y=0$: the x-axis except $(0,0)$.

$k=1$: $\dfrac{4y}{x^2+y^2} = 1 \rightsquigarrow x^2+y^2 = 4y$.

$\rightsquigarrow x^2+y^2-4y=0 \rightsquigarrow x^2+y^2-4y+4=4$

$\rightsquigarrow x^2+(y-2)^2=4$: a circle

$k=2$: $\dfrac{\overset{2}{\cancel{4}}y}{x^2+y^2} = \overset{1}{\cancel{2}} \rightsquigarrow x^2+y^2 = 2y$

$\rightsquigarrow x^2+y^2-2y=0 \rightsquigarrow x^2+y^2-2y+1=1$

$\rightsquigarrow x^2+(y-1)^2=1$: a circle.

$k=-1$: $\dfrac{4y}{x^2+y^2} = -1 \rightsquigarrow -(x^2+y^2) = 4y$

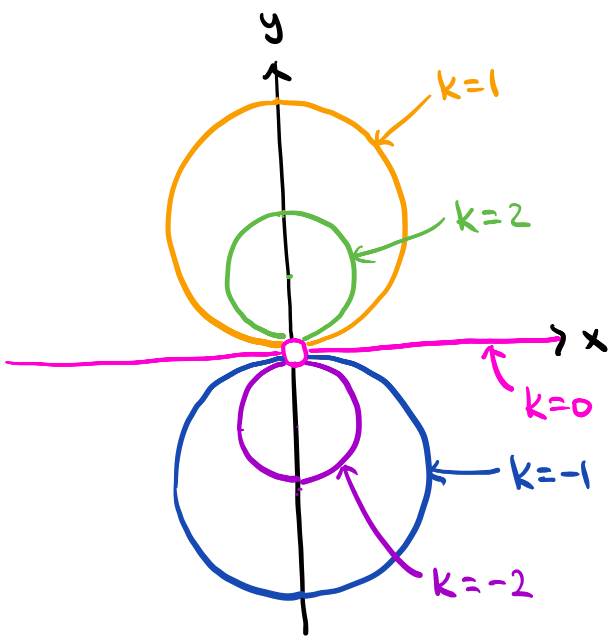
$\rightsquigarrow x^2+y^2+4y=0 \rightsquigarrow x^2+y^2+4y+4=4$

$\rightsquigarrow x^2+(y+2)^2=4$: a circle

$k=-2$: $\dfrac{\cancel{4}^{2}y}{x^2+y^2} = -\cancel{2}^{1} \rightsquigarrow -(x^2+y^2) = 2y$

$\rightsquigarrow x^2+y^2+2y = 0 \rightsquigarrow x^2+y^2+2y+1 = 1$

$\rightsquigarrow x^2+(y+1)^2 = 1$: a circle.



<u>Note</u> Two level curves at different levels cannot cross each other.

(If they do, then the intersection point should have two different function values)